Asymptotic Description of Radiating Flow near Stagnation Point

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The correct asymptotic behavior of an inviscid radiating gas flow near the stagnation point of a blunt body in the weakly radiating and weakly absorbing limits is given. While reflecting different physical situations, the two limits are shown to imply similar singular perturbation behavior which has been overcome through use of the method of matched asymptotic expansions. Comparison of these results with those obtained using the PLK asymptotic technique indicate the failure of the latter to correctly overcome the singular behavior.

Nomenclature

E_n = nth order exponential integral function
h = enthalpy
m = exponent in variation of volumetric absorption coefficient with temperature
n = exponent in variation of enthalpy with temperature
p = pressure
qR = radiative heat flux
T = temperature
u, U = velocity
x = coordinate normal to body in related radiationless flow
x_n = nth order coordinate straining function
X, Y = physical coordinates normal and parallel to body, respectively
z = independent variable of PLK application
α = $(2m + n)/2n$
β = $4/n$
Γ = radiation convection energy parameter
Δ = nonradiating shock detachment distance
Δ_{RAD} = radiating shock detachment distance
ϵ = small parameter
ϵ_w = wall emissivity
ζ = stretched inner variable
κ = volumetric absorption coefficient
ρ = density
τ = optical depth
τ_s = reference optical depth
ξ = stretched inner variable

Subscripts

MAE = matched asymptotic expansion
PLK = Poincaré-Lighthill-Kuo
W = wall
wg = value in gas adjacent to wall
s = shock
$*$ = uniformly valid

Superscripts

C = composite solution
i, I = inner solution
ϑ = outer solution

Introduction

THE role of radiative energy transfer in high-velocity reentry has been of considerable interest owing to the enhanced heating rates it effects.[1,2] In addition to implying local nonadiabaticity through emission, radiative transfer, as a result of the global nature of the absorption process, implies a set of descriptive conservation equations which are of a nonlinear integro-differential form and whose general solution requires elaborate numerical efforts.[2,3]

Attempts to elucidate the basic effects of radiative transfer in high-speed, high-temperature flows through use of simplifying approximations have been in evidence. Preliminary estimates assuming the radiative energy transfer to be small (usually in comparison with the convective energy flux) were given by Goulard.[4] Such a procedure, valid for velocities below, say, 11 km/sec, reduces the governing equations to much more tractable form. The results yield decreased temperature levels in the flowfield due to the energy loss by radiative transfer with a commensurate decrease in the radiative heat transfer. In the case of stagnation point flow, the approximate temperature profile is logarithmically divergent near the stagnation point. Attempts have been made to correct this singular behavior through use of the PLK (Poincaré-Lighthill-Kuo) asymptotic technique[5] and through the use of a local temperature approximation.[6] Another simplifying approximation which has been of some interest is the weakly absorbing (often referred to as optically thin or emission-dominated) limit in which the effects of self-absorption are neglected. Here the governing equations reduce to a purely differential form and the effects of radiative transfer are easily evaluated. Goulard[4] was the first to show the resulting zero temperature of the gas at the wall in the optically thin stagnation point flow case. Jischke[7] has recently shown that this behavior is the result of the nonuniform validity of the optically thin limit and can be overcome through use of the method of matched asymptotic expansions. Burggraf[8] has indicated that this singular behavior can be removed by consideration of the viscous boundary layer near the stagnation point.

It is the purpose of the present paper to show that both of the aforementioned approximations (e.g. optically thin and weakly radiating) imply singular perturbation behavior near the stagnation point itself which, although having different physical interpretations, are of a similar mathematical nature. Use of the method of matched asymptotic expansions allows one to overcome these singular perturbation difficulties. Also, comparison of these results with those obtained using the PLK asymptotic technique indicate that in the present problem, the PLK method fails to correctly resolve singular behavior.

Analysis

Consider the flow of an inviscid, radiating gas in the stagnation region of a blunt body. Assume the temperature in the stagnation region is given by

$$T(X,Y) = T(X) + \vartheta(Y^2)$$

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(X and Y are coordinates normal to and parallel to the body, respectively.) and that behind the strong bow shock wave $p \gg \rho U^2/2$ and $h \gg U^2/2$. This implies a constant pressure shock layer. With these approximations, conservation of energy gives

$$\rho u \frac{dh}{dX} = -\kappa \frac{dq^R}{d\tau} =$$
$$-2\sigma\kappa \left[2T^4 - \epsilon_W T_W{}^4 E_2(\tau) - \int_0^{\tau_{\text{shock}}} T^4 E_1(|\tau - t|)dt\right] \quad (1)$$

where κ is the volumetric absorption coefficient, q^R, is the radiative heat flux, and τ, the optical depth, is given by

$$\tau = \int_0^X \kappa dX \quad (2)$$

with $\tau_{\text{shock}} = \tau(X_{\text{shock}})$. E_n is the nth-order exponential integral function. The one-dimensional "slab" form for the radiative flux divergence neglecting upstream absorption, i.e., Eq. (1), is valid when the shock detachment distance is small relative to the shock radius of curvature and the freestream temperature is small. Following Goulard,[4] we relate the physical coordinate X to coordinate x of the corresponding radiationless case by

$$\rho^{1/2}dX = \rho_s{}^{1/2}dx \quad (3)$$

and assume the velocity component u parallel to the axis of the body is simply related to the velocity component U in the corresponding radiationless (constant density) case by

$$\rho u(X) = \rho_s U(x) = -\rho_s|U_s|\, x/\Delta \quad (4)$$

where Δ is the nonradiating shock detachment distance. Thus in nondimensional form, Eq. (1) becomes

$$\bar{x}\frac{d\bar{h}}{d\bar{x}} = \Gamma\bar{\kappa}\bar{\rho}^{-1/2}\left\{2\bar{T}^4 - \epsilon_W \bar{T}_W{}^4 E_2\left(\tau_s \int_0^{\bar{x}} \bar{\kappa}\bar{\rho}^{-1/2}d\bar{x}\right) - \tau_s \int_0^1 \bar{T}^4\bar{\kappa}\bar{\rho}^{-1/2}E_1\left(\tau_s\left|\int_{\bar{x}}^{\bar{x}'} \bar{\kappa}\bar{\rho}^{-1/2}d\bar{x}''\right|\right)d\bar{x}'\right\} \quad (5)$$

where the bars imply nondimensional quantities. Here

$$\bar{h} = h/h_s,\ \bar{\rho} = \rho/\rho_s,\ \bar{T} = T/T_s,\ \bar{\kappa} = \kappa/\kappa_s$$
$$\bar{x} = x/\Delta,\ \tau_s = \kappa_s\Delta,\ \Gamma = (2\sigma T_s{}^4/\rho_s u_s h_s)\tau_s$$

Γ is the radiation convection energy parameter (inverse Boltzmann number) and indicates the relative importance of energy transfer by radiation and convection. τ_s is the reference optical depth (Bouguer number) and is a measure of the importance of self-absorption (as compared with emission). The appropriate boundary condition for Eq. (5) is

$$\bar{h}(\bar{x} = 1) = 1$$

From Eq. (3), at $\bar{x} = 1$

$$\bar{X}_s = \frac{X_s}{\Delta} = \frac{\Delta_{\text{RAD}}}{\Delta} = \int_0^1 \bar{\rho}^{-1/2}d\bar{x} \quad (6)$$

We now assume the temperature, density, and volumetric absorption coefficient depend upon the enthalpy in the following approximate but reasonable way

$$\bar{T} = \bar{h}^{1/n},\ \bar{\rho} = \bar{h}^{-1},\ \bar{\kappa} = \bar{T}^m = \bar{h}^{m/n}$$

Eq. (5) then becomes, dropping the bar notation

$$x\frac{dh}{dx} = \Gamma h^\alpha\left\{2h^\beta - \epsilon_W T_W{}^4 E_2\left(\tau_s\int_0^x h^\alpha dx'\right) - \tau_s\int_0^1 h^{\alpha+\beta}E_1\left(\tau_s\left|\int_x^{x'} h^\alpha dx''\right|\right)dx'\right\} \quad (7)$$

where α and β are related to m and n by

$$\alpha = (2m + n)/2n,\ \beta = 4/n$$

Note that the nonlinear integro-differential nature of the problem is retained in this simplified model.

A. Γ Small

The limit of small Γ has been of interest as it corresponds to the case of a weakly radiating flow and thus allows one to obtain a first estimate of the effect of radiative transfer on the flowfield. Expanding the enthalpy h in powers of Γ,

$$h = h_0 + \Gamma h_1 + \ldots \quad (8)$$

gives

$$h_0(x) = 1$$
$$h_1(x) = \int_1^x \{E_2(\tau_s x) + E_2[\tau_s(1 - x)] - \epsilon_W T_W{}^4 E_2(\tau_s x)\}\frac{dx}{x}$$

Note that h_1 is logarithmically singular near $x = 0$. That is

$$h_1 \sim \ln x \text{ as } x \to 0$$

and thus a perturbation about $\Gamma = 0$ is singular. The source of this difficulty is clear. For any value of Γ not equal to zero, conservation of energy at a stagnation point implies radiative equilibrium, $(\nabla\cdot\mathbf{q}^R) = 0$. However, evaluation of the radiative flux divergence on the basis of a nonradiating solution does not necessarily satisfy this condition and indeed implies a finite nonzero value of $(\nabla\cdot\mathbf{q}^R)$ at $x = 0$. In order for a consistent convective energy flux to balance this nonzero radiative flux divergence as the velocity vanishes, the enthalpy gradient must become infinite (like $1/x$) which implies the logarithmic divergence for the enthalpy itself. Comparing the zeroth and first order terms, it is apparent that the regular perturbation solution is no longer valid when

$$\Gamma \ln x = \vartheta(1)$$

Thus using the method of matched asymptotic expansions with the "outer" solution corresponding to the above regular perturbation solution, we introduce a stretched "inner" variable ζ where

$$\zeta = -\Gamma \ln x \quad (9)$$

Equation (7) written in terms of this inner variable then becomes

$$\frac{dh^i}{d\zeta} = -h^{i\alpha}\left\{2h^{i\beta} - \epsilon_W T_W{}^4 E_2\left(\tau_s\int_0^{e^{-\zeta/\Gamma}} h^\alpha dx'\right) - \tau_s\int_0^1 h^{\alpha+\beta}E_1\left(\tau_s\left|\int_{e^{-\zeta/\Gamma}}^{x'} h^\alpha dx''\right|\right)dx'\right\} \quad (10)$$

where the superscript i implies inner solution. We expand the enthalpy in powers of Γ with ζ fixed,

$$h^i = h_0{}^i(\zeta) + \Gamma h_1{}^i(\zeta) + \ldots$$

Eq. (10) becomes, to lowest order,

$$\frac{dh_0{}^i}{d\zeta} = -h_0{}^{i\alpha}\left\{2h_0{}^{i\beta} - \epsilon_W T_W{}^4 - \tau_s\int_0^1 h^{\alpha+\beta} \times E_1\left(\tau_s\int_0^{x'} h^\alpha dx''\right)dx'\right\} \quad (11)$$

Note however that due to the exponentially small extent of this inner region, the absorption integral, being an integral over the entire shock layer, is determined to within an exponentially small error by the outer (regular perturbation) solution. More formally, assuming a composite solution exists, an asymptotically correct value of the absorption integral is given by

$$\int_0^1 f dx \sim \int_0^1 f^c dx = \int_0^1 \sum_{n=0}^{\infty} \Gamma^n(f_n{}^\vartheta + f_n{}^i - f_n{}^{i\vartheta})\,dx'$$

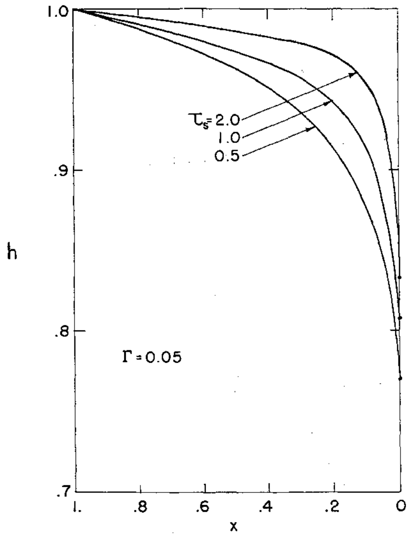

Fig. 1 Shock-layer enthalpy profile (Γ small).

Here f is a shorthand notation for

$$h^{\alpha+\beta}E_1\left(\tau_s\left|\int_0^x h^\alpha dx'\right|\right)$$

and the superscripts ϑ and c stand for outer (regular perturbation) and composite solution, respectively. Making use of the fact that f_n^i and $f_n^{i\vartheta}$ coincide in the outer region

$$\int_0^1 fdx' \sim \int_0^1 \sum_{n=0}^{\infty} \Gamma^n f_n^\vartheta dx' + \int_0^{g(\Gamma)} \sum_{n=0}^{\infty} \Gamma^n(f_n^i - f_n^{i\vartheta})dx'$$

where $g(\Gamma)$ is a small number denoting a value of x in the overlap region whose existence is assured by the assumption of the existence of a composite solution. As g must satisfy

$$\lim_{\Gamma\to 0} g(\Gamma) = 0, \quad \lim_{\Gamma\to 0} \frac{e^{-1/\Gamma}}{g(\Gamma)} = 0$$

[A suitable choice would be $g(\Gamma) = e^{-1/\Gamma}/\Gamma$] it follows that to any algebraic order in Γ,

$$\int_0^1 fdx \sim \sum_{n=0}^{\infty} \Gamma^n \int_0^1 f_n^\vartheta dx \tag{12}$$

Thus global integrated quantities such as the shock detachment distance and radiative heat transfer are determined by the outer (regular perturbation) solution.

This result appears to be of some general importance. Specifically, for problems in radiative gas dynamics involving singular perturbation behavior with associated "boundary layers," the boundary layer will be a region of constant absorption with the absorption level being determined by the "inviscid," regular perturbation solution. The only restrictions would appear to be that the boundary layer Bouguer number be small in comparison to unity and the emissive power be integrable in the boundary layer.

Recall that the outer solution for the enthalpy is of the form

$$h^\vartheta = 1 + \Gamma h_1^\vartheta + \ldots$$

It follows that the absorption integral is given, in the inner region, by

$$\tau_s \int_0^1 h^{\vartheta\alpha+\beta} E_1\left(\tau_s \int_0^{x'} h^{\vartheta\alpha} dx''\right) dx' = 1 - E_2(\tau_s) + \vartheta(\Gamma \ln \Gamma)$$

and thus, to lowest order, the inner solution follows from

$$dh_0^i/d\zeta = -h_0^{i\alpha}\{2h_0^{i\beta} - \mathcal{J}_0\} \tag{13}$$

where $\mathcal{J}_0$, the lowest order net absorption term, is given by

$$\mathcal{J}_0 = \epsilon_W T_W^4 + 1 - E_2(\tau_s)$$

Note that Eq. (13) yields a finite value for the gas enthalpy at the wall, h_{wg}

$$h_{wg} = h(\zeta \to \infty) = (\mathcal{J}_0/2)^{1/\beta} \tag{14}$$

which is consistent with the condition of radiative equilibrium. Thus, physically, the matched asymptotic expansion result states that, to lowest order, as one approaches the stagnation point, the nature of the energy conservation changes character from a balance between convection and the (small) net radiative energy transfer (emission minus absorption) to a balance between convection, emission, and absorption—all being of order one in this inner region. Further, this region, being of such small extent, is a "constant absorption" region with the constant value of the absorption integral being determined, to lowest order, by the nonradiating profile. From Eq. (13), the lowest-order inner solution can be written as a quadrature,

$$\zeta = -\int_1^{h_0^i} \frac{y^{-\alpha}}{2y^\beta - \mathcal{J}_0} dy + \frac{\Gamma\tau_s}{1 + E_2(\tau_s) - \epsilon_W T_W^4} \times \int_0^1 \ln x\{E_1[\tau_s(1-x)] - E_1(\tau_s x) + \epsilon_W T_W^4 E_1(\tau_s x)\} dx \tag{15}$$

where the lower limit follows from the matching condition. The composite, uniformly valid, solution is

$$\begin{aligned} h^c &= h^i + h^\vartheta - h^{i\vartheta} \\ &= h_0^i + \Gamma\{E_2(\tau_s x) + E_2[\tau_s(1-x)] + \\ &\quad \epsilon_W T_W^4[1 - E_2(\tau_s x)] - 1 - E_2(\tau_s)\} \ln x - \\ &\quad \Gamma\tau_s \int_0^x \ln x\{E_1[\tau_s(1-x)] - E_1(\tau_s x) + \epsilon_W T_W^4 \times \\ &\quad E_1(\tau_s x)\} dx + \vartheta(\Gamma \ln \Gamma) \end{aligned} \tag{16}$$

with the lowest-order inner solution following from Eq. (15). Typical results are shown in Figs. 1–3 where the shock-layer enthalpy profile, shock detachment distance, radiative heat transfer, and gas enthalpy at the wall are given. As shown, radiative transfer, being an energy loss mechanism, leads to decreased values of the enthalpy which implies increased values of the density and hence smaller shock detachment distances. These lower enthalpy levels, as shown, also lead to reduced heating rates.

B. τ_s Small

As mentioned in the introduction, the weakly absorbing (optically thin) limit has also been of interest. While not restricting the level of radiative transfer, one argues in this limiting situation that the photon mean free path κ^{-1} is much larger than the scale of the problem—here the shock detachment distance—and thus the effects of self-absorption are negligible. For example, for a 1-ft sphere moving at 40,000

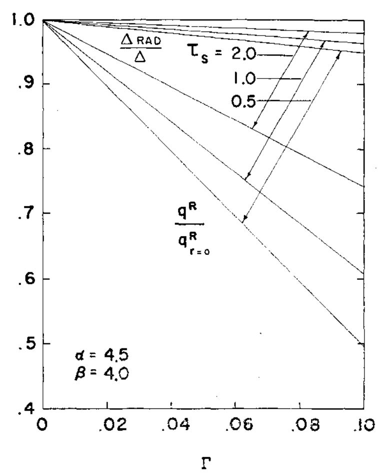

Fig. 2 Effect of radiative transfer on the shock detachment distance and heat transfer to the body (Γ small).

fps at 150,000 ft altitude, τ_s is approximately 0.1. This limit is analogous to the free molecular limit in rarefied gas flows.

To simplify the analysis, let us use the approximate representation of $E_1(x)$,

$$E_1(x) \approx (3)^{1/2}e^{-(3)^{1/2}x}$$

corresponding to the differential approximation of Cheng[9] and Traugott.[10] As shown in Ref. 3, such an approximation does not alter the essential points of the discussion. Thus, we expand h in powers of τ_s

$$h = h_o + \tau_s h_1 + \ldots$$

Substituting this expansion in Eq. (7) makes clear that the nature of the perturbation scheme depends quite dramatically on the value of the wall emissive power, $\epsilon_W T_W^4$. In particular, for $\epsilon_W T_W^4$ identically zero, the perturbation is singular near $x = 0$ whereas for $\epsilon_W T_W^4$ nonzero, no such difficulties arise. To see this, consider the case for $\epsilon_W T_W^4$ identically zero. Then one can show quite readily that

$$h_0 = [1 - 2\Gamma(\alpha + \beta - 1)\ln x]^{-1/(\alpha+\beta-1)}$$

$$h_1 = \frac{J_1}{2(\alpha + \beta - 1)} \times$$
$$[\{1 - 2\Gamma(\alpha+\beta-1)\ln x\}^{(\beta-1)/(\alpha+\beta-1)} - \{1 - 2\Gamma(\alpha+\beta-1)\ln x\}^{-(\alpha+\beta)/(\alpha+\beta-1)}] \quad (17)$$

where the absorption integral, J_1 is given by

$$J_1 = (3)^{1/2}\int_0^1 h_0^{\alpha+\beta}dx = (3)^{1/2}\int_0^1 [1 - 2\Gamma(\alpha+\beta-1)\ln x]^{-(\alpha+\beta)/(\alpha+\beta-1)}dx \quad (18)$$

Note that, to lowest order, the gas enthalpy at the wall vanishes. Also, h_1 diverges logarithmically as x approaches zero (the fact that the divergence is logarithmic is a reflection of the nonlinear form of the radiative emission). Thus a regular perturbation scheme based upon the smallness of τ_s is not uniformly valid. Such an approach becomes invalid near the stagnation point where the complete energy equation implies radiative equilibrium. As h_1 is proportional to the radiation absorption integral J_1 (which is evaluated on the basis of the nonabsorbing profile), one can conclude that a gas particle as it approaches the stagnation point has radiated away so much energy that local emission becomes of the same (small) order as global absorption. Therefore, sufficiently close to the stagnation point, self-absorption cannot be neglected. The uniformly valid solution will now be obtained using the method of matched asymptotic expansions in much the same spirit as that for Γ small.

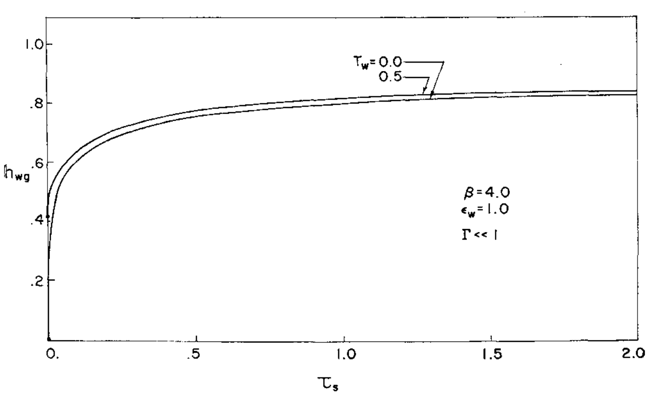

Fig. 3 Effect of radiative transfer on gas enthalpy at the wall (Γ small).

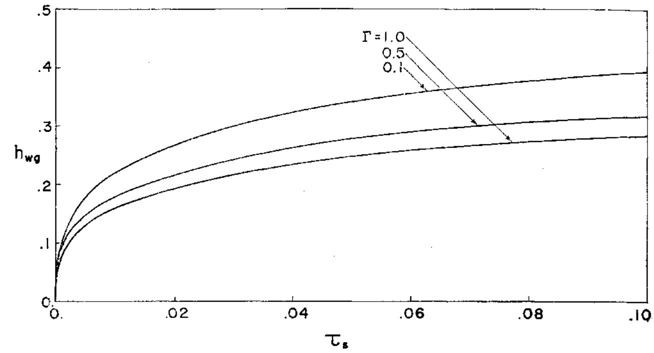

Fig. 4 Effect of radiative transfer on gas enthalpy at the wall (τ_s small, T_w = 0).

We introduce the stretched "inner" independent and dependent variables ζ and h^I

$$\zeta = -\Gamma\tau_s^{(\alpha+\beta-1)/\beta}\ln x$$
$$h^i = \tau_s^{1/\beta}h^I \quad (19)$$

(h^I being of order one for ζ of order one). h^I follows from

$$dh^I/d\zeta = -h^{I\alpha}\{2h^{I\beta} - \mathcal{J}\} \quad (20)$$

where here $\mathcal{J}$, the absorption integral, is given by

$$\mathcal{J} = (3)^{1/2}\int_0^1 h^{\alpha+\beta} \times \exp\left(-(3)^{1/2}\tau_s\left|\int_{e^{-\zeta/\Gamma\tau_s(\alpha+\beta-1)/\beta}}^{x'} h^\alpha dx''\right|\right)dx'$$

As earlier, owing to the exponentially small extent of the inner region, this absorption integral can be evaluated, to any algebraic order in τ_s using the "outer" enthalpy profile. Thus we expand the enthalpy in the inner region in powers of the reference optical depth τ_s,

$$h^I = h_0^I + \tau_s h_1^I + \ldots$$

Equation (20) then becomes, to lowest order,

$$dh_0^I/d\zeta = -h_0^{I\alpha}\{2h_0^{I\beta} - \mathcal{J}_0\} \quad (21)$$

where $\mathcal{J}_0$, the lowest-order absorption integral, can be shown to be equal to J_1. Note that h_0^I satisfies an equation which is quite similar to the lowest-order inner solution in the weakly radiating case [compare Eq. (21) and (13)]—in both cases, the inner region is a *constant absorption* region. However, physically, this result, Eq. (21), implies that sufficiently close to the stagnation point, the local energy balance is between convection, local emission, and *global* absorption (as opposed to a balance between local emission and convection as is appropriate in the rest of the shock layer) and this global absorption term is given to lowest order by the nonabsorbing (outer) profile. Writing the lowest order inner solution as a quadrature,

$$\zeta = -\int_{h_0^I}^{\tau_s^{(\alpha+\beta-1)/\beta}} \frac{y^{-\alpha}}{(2y^\beta - \mathcal{J}_0)}dy \quad (22)$$

where the upper limit follows from matching. Note that to lowest order, the composite solution is just the inner solution as the outer limit of the inner solution is the outer solution. The gas enthalpy at the wall, h_{wg} is given by

$$h_{wg} = (\tau_s\mathcal{J}_0/2)^{1/\beta} \quad (23)$$

from the condition of radiative equilibrium at the stagnation point. Typical numerical results obtained from this relation are shown in Fig. 4 for $\beta = 4$ and $\alpha = 9/2$.

Turning now to the case of nonzero wall emissive power, the lowest order enthalpy profile (in an expansion in powers of τ_s)

follows from

$$\Gamma \ln x = \int_1^{h_0} \frac{y^{-\alpha}}{2y^\beta - \epsilon_W T_W{}^4} dy \tag{24}$$

and for x near zero (close to the stagnation point)

$$h_0 = \left(\frac{\epsilon_W T_W{}^4}{2}\right)^{1/\beta} + a_0 x^{2\beta\Gamma(\epsilon_w T_w{}^4/2)^{(\alpha+\beta-1)/\beta}} + \ldots$$

Similarly, for x near zero, the first-order solution is of the form

$$h_1 = \frac{(3)^{1/2}}{2\beta}\left(\frac{\epsilon_W T_W{}^4}{2}\right)^{-1+(1/\beta)} \times \int_0^1 h_0{}^{\alpha+\beta} dx + a_1 x^{2\beta\Gamma(\epsilon_w T_w{}^4/2)^{(\alpha+\beta-1)/\beta}} + \ldots$$

Here the a_i are constants. The essential point to note here is that the perturbation about $\tau_s = 0$ is nonsingular for nonvanishing wall emissive powers and becomes singular in the limit of the wall emission power approaching zero $[h_1 \sim (\epsilon_W T_W{}^4)^{-1+(1/\beta)}]$. This nonsingular behavior results since basic solution ($\tau_s = 0$) satisfies the condition of radiative equilibrium at the stagnation point through a physically correct balance between emission and nonzero absorption implying the effect of self-absorption to be uniformly small for τ_s sufficiently small.

Analytical expressions for the enthalpy profile in this nonabsorbing limit can be obtained only for special values of α and β. For example, for $\alpha = 1 - \beta$ (e.g., $m = -3.5$, $n = 1$ corresponds to extremely high-temperature air,[10] say above 10^5°K

$$h_0 = \{\tfrac{1}{2}\epsilon_W T_W{}^4 + (1 - \tfrac{1}{2}\epsilon_W T_W{}^4) x^{2\beta\Gamma}\}^{1/\beta} \tag{25}$$

C. PLK Method

Attempts to overcome the singular perturbation difficulties associated with small Γ or τ_s using the PLK (Poincaré-Lighthill-Kuo) method† have been put forth.[5] For purposes of comparison with results obtained using the method of matched asymptotic expansions, let us consider the weakly radiating flow case as an example. According to the PLK technique (see, e.g., Van Dyke[11] or Pritulo[12]) one obtains the uniformly valid solution by perturbing both the independent and dependent variables. Pritulo has shown that such a procedure implies the following general relationship between the uniformly valid solution (indicated by the subscript *) and the regular perturbation (singular) solution,

$$h_{0*}(z) = h_0(z) \tag{26}$$

$$h_{1*}(z) = h_1(z) + x_1(z) dh_0/dz \tag{27}$$

$$h_{2*}(z) = h_2(z) + x_2(z) dh_0/dz + x_1(z) dh_1/dz + x_1{}^2(z) d^2h_0/dz^2 \tag{28}$$

etc., with x given by

$$x = x(z) = z + \Gamma x_1(z) + \Gamma^2 x_2(z) + \ldots \tag{29}$$

The coordinate straining functions x_n are chosen so that h_n is no more singular than h_{n-1}. Recalling that h_0 is constant, the uniformly valid solution follows from

$$h_*(z) = h_0(z) + \Gamma h_1(z) + \vartheta(\Gamma^3) \tag{30}$$

where

$$x = z - \frac{\Gamma h_2(z)}{(dh_1/dz)} + \vartheta(\Gamma^2) \tag{31}$$

with x_1 having been determined from setting h_{2*} to zero.

† Sometimes referred to as the method of strained coordinates, Lighthill's method, or the method of perturbation of coordinates.

Comparing results obtained using the above PLK technique with those obtained using the method of matched asymptotic expansions, we find that the two approaches give for the gas enthalpy at the wall

$$h_{wg_{\text{MAE}}} = \left(\frac{\epsilon_W T_W{}^4}{2}\right)^{1/\beta} + \vartheta(\Gamma) \tag{32}$$

$$h_{wg_{\text{PLK}}} = 1 - (\Gamma)^{1/2} \frac{(2 - \epsilon_W T_W{}^4)}{[\alpha + \beta - \alpha(\epsilon_W T_W{}^4/2)]^{1/2}} + \vartheta(\Gamma) \tag{33}$$

Here, for simplicity, we have assumed $\tau_s \ll 1$. As the matched asymptotic expansions result is known to give the physically correct result of radiative equilibrium at the stagnation point ($x = 0$), it would appear that the PLK technique gives an erroneous answer. A similar conclusion holds for the optically thin perturbation case. Further substantiation of this conclusion follows from consideration of a rather trivial (but instructive) model equation

$$xdy/dx = \epsilon(y^2 - \alpha^2),\ y(1) = 1,\ \epsilon \ll 1$$

for which exact analytical results can be obtained.

Thus one is forced to conclude that the PLK technique is not applicable to ordinary differential equations of the form considered previously. While it is difficult and sometimes dangerous to attempt to draw general conclusions in situations such as this, it should be noted that the equations considered all have nodal point behavior near the singular point ($x = 0$) and that the correct description near the singular point follows from a *nonlinear* equation.

Conclusions

It has been shown that perturbing a nonradiating flow for small radiation effects implies singular perturbation behavior. Also, perturbing a nonabsorbing flow for small self-absorption effects is singular if, and only if, the wall emissive power is zero. Use of the method of matched asymptotic expansions allows one to overcome these singular perturbation difficulties. In both cases the region of singular behavior (i.e., the inner region near the stagnation point) is a region of constant absorption with variable emission and the absorption level is determined by the regular perturbation (outer) solution. It would appear that these conclusions are also valid for radiative transfer models which are more realistic than the gray gas model employed here. Lastly, application of the PLK asymptotic technique fails to correctly resolve the singular behavior as has been assumed elsewhere.

References

[1] Cheng, P. and Vincenti, W., "Inviscid Radiating Flow Over a Blunt Body," *Journal of Fluid Mechanics*, Vol. 27, No. 4, 1967, pp. 625–646.

[2] Finkleman, D., "Self-Consistent Solution of Nonlinear Problems in Unsteady Radiation Gasdynamics," AIAA Paper 68-163, New York, 1968.

[3] Jischke, M. and Baron, J. R., "Application of the Method of Parametric Differentiation to Radiative Gasdynamics," *AIAA Journal*, Vol. 7, No. 7, July 1969, pp. 1326–1335.

[4] Goulard, R., "Preliminary Estimates of Radiative Transfer Effects on Detached Shock Layers," *AIAA Journal*, Vol. 2, No. 3, March 1964, pp. 492–502.

[5] Olstad, W., "Stagnation—Point Solutions for an Inviscid, Radiating Shock Layer," *1965 Heat Transfer and Fluid Mechanics Institute*, Stanford Univ. Press, 1965, p. 138.

[6] Jischke, M., "Optically Thin Stagnation Point Flow," *AIAA Journal*, Vol. 6, No. 11, Nov. 1968, pp. 2219–2222.

[7] Thomas, P., "Transparency Assumption in Hypersonic Radiative Gasdynamics," *AIAA Journal*, Vol. 3, No. 8, Aug. 1965, pp. 1401–1408.

[8] Burggraf, O., "Asymptotic Solution for the Viscous Radiating Shock Layer," *AIAA Journal*, Vol. 4, No. 10, Oct. 1966, pp. 1725–1733.

[9] Cheng, P., "Dynamics of a Radiating Gas with Application to Flow over a Wavy Wall," *AIAA Journal*, Vol. 4, No. 2, Feb. 1966, pp. 238–245.

[10] Traugott, S., "A Differential Approximation for Radiative Transfer with Application to Normal Shock Structure," Research Report 34, Dec. 1962, Martin Co.

[11] Van Dyke, M., *Perturbation Methods in Fluid Mechanics*, Academic Press, New York, 1964.

[12] Pritulo, M., "On the Determination of Uniformly Accurate Solutions of Differential Equations by the Method of Perturbation of Coordinates," *Prikladnaza Matamatika e Mechanika*, Tom XXVI, No. 3, 1962, pp. 444–448.

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Temperature Distribution and Effectiveness of a Two-Dimensional Radiating and Convecting Circular Fin

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An analysis is made of the heat-transfer characteristics of a circular fin dissipating heat from its surface by convection and radiation. The temperature is assumed uniform along the base of the fin and constant physical and surface properties are assumed. There is radiant interaction between the fin and its base. Two separate situations are considered. In the first situation heat transfer from the end of the fin is neglected. Solution of the linear conduction equation with nonlinear boundary conditions has been obtained by a least-squares fit method, and also by the finite difference method and the results compared. Results are presented for a wide range of environmental conditions and physical and surface properties of the fin. In the second situation, heat transfer from the end of the fin is also included in the analysis. The solution for the second situation is obtained by a finite-difference procedure only. It is shown that neglecting heat transfer from the end is a good approximation for long fins or for fins of high thermal conductivity material.

Nomenclature

a = radius of circular fin, ft
A = $\epsilon_1\sigma a T_0^3/k$, radiation-conduction parameter, dimensionless
F = configuration factor, dimensionless
h = heat-transfer coefficient, Btu/hr ft² °R
k = thermal conductivity of fin, Btu/hr ft °R
l = fin length, ft
L = l/a, dimensionless fin length
N = ha/k, convection parameter, (Biot number), dimensionless
Q = rate of heat loss from fin, Btu/hr
r,z = cylindrical coordinates for fin, ft
R,Z = r/a, z/a, dimensionless cylindrical coordinates
r_2 = fin base radius, ft
T = absolute temperature, °R
T_0 = fin base temperature, °R
T_∞ = fluid bulk temperature, °R
T^* = effective radiation environment temperature, °R
α = coefficient of absorptivity, dimensionless
ϵ = coefficient of emissivity, dimensionless
σ = Stefan-Boltzmann constant, 0.1714×10^{-8} Btu/hr ft² °R⁴
η = fin effectiveness
β = r_2/a, dimensionless
λ = $(T - T_\theta)/T_0$, dimensionless temperature at any point in the fin
λ_∞ = T_∞/T_0, dimensionless fluid bulk temperature
λ^* = T^*/T_0, dimensionless effective radiation environment temperature

Subscripts

1 = fin surface
2 = base surface
∞ = fluid bulk

Superscripts

* = effective radiation environment

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Introduction

THE solar energy absorbed by a space vehicle by direct incidence or reflection from planets and the energy generated by electronic instruments in the vehicle itself have to be dissipated away to the surroundings. To limit the large amounts of heat-transfer area required on the space vehicle, fins are extensively used.

Although several studies have been made on the steady-state heat transfer for radiating fins, most of the work has been for the one-dimensional model. Shouman[1] has obtained an exact general solution for a constant cross-sectional area fin. However, he considered a one-dimensional model with no fin-to-base interaction. Liu[2] had earlier developed an exact solution for the rectangular profile fin. Lieblein[3] obtained a finite difference solution for the rectangular fin while Bartas and Sellars[4] determined the fin effectiveness for one-dimensional heat flow in rectangular fins by numerical methods. Sparrow and Eckert[5] considered the effects of mutual irradiation occurring between a fin and its adjoining base surfaces. Sparrow, Eckert, and Irvine[6] used numerical iterative methods to analyze the effectiveness of plane radiating fins with mutual irradiation. Chambers and Somers[7] numerically determined the fin efficiency for a flat annular fin. Sparrow, Miller, and Jonsson[8] used finite difference methods to calculate the fin effectiveness for one-dimensional heat flow in annular fins with mutual irradiation between the black radiator elements. Recently, Sparrow and Nie-